Number-conserving interacting fermion models with exact topological superconducting ground states

Zhiyuan Wang,[1,2,*] Youjiang Xu,[2] Han Pu,[2,3,4] and Kaden R. A. Hazzard[2,4]
[1]*School of Physics, Peking University, Beijing 100871, China*
[2]*Department of Physics and Astronomy, Rice University, Houston, Texas 77005, USA*
[3]*Center for Cold Atom Physics, Chinese Academy of Sciences, Wuhan 430071, China*
[4]*Rice Center for Quantum Materials, Rice University, Houston, Texas 77005, USA*
(Received 3 March 2017; revised manuscript received 22 August 2017; published 7 September 2017)

We present a method to construct number-conserving Hamiltonians whose ground states exactly reproduce an arbitrarily chosen BCS-type mean-field state. Such parent Hamiltonians can be constructed not only for the usual s-wave BCS state, but also for more exotic states of this form, including the ground states of Kitaev wires and two-dimensional topological superconductors. This method leads to infinite families of locally interacting fermion models with exact topological superconducting ground states. After explaining the general technique, we apply this method to construct two specific classes of models. The first one is a one-dimensional double wire lattice model with Majorana-like degenerate ground states. The second one is a two-dimensional $p_x + ip_y$ superconducting model, where we also obtain analytic expressions for topologically degenerate ground states in the presence of vortices. Our models may provide a deeper conceptual understanding of how Majorana zero modes could emerge in condensed matter systems, as well as inspire novel routes to realize them in experiment.

DOI: 10.1103/PhysRevB.96.115110

I. INTRODUCTION

Topological superconductors have become an active research area in condensed matter and cold atom physics [1,2]. They provide examples of topological phases that have been classified systematically [3–9]. More practically, the non-Abelian statistics [10–12] of Majorana zero modes and the robustness of degenerate ground states against local perturbations have made topological superconductors components of promising architectures for fault-tolerant quantum computation [13]. Experimental signatures [14–17] of Majorana zero modes and topological superconductivity in solid state systems call for more realistic theoretical descriptions of the relevant physics, for example the effect of interactions.

Most of the theoretical research in this area has begun with noninteracting mean-field Hamiltonians with an effective p-wave pairing term, from which one obtains topologically protected degenerate ground states and Majorana zero modes. In nature, however, sizable interactions typically challenge the validity of mean-field theory. It is therefore important to understand better the criteria for the aforementioned topological phenomena to persist in the interacting case. Moreover, the mean-field approximation breaks the number conservation, which obscures the connection to realistic, number-conserving systems.

Understanding the interplay of topology, number conservation, and interactions is challenging because the interactions usually prevent exact solution by analytic or numerical techniques. In one dimension, special tools are available, and progress has been made using bosonization [18–20] and numerical methods [density-matrix renormalization group (DMRG)] [21]. Exactly solvable models in this area are still rare [22], and the two number-conserving Majorana models [23,24] that have been proposed are one dimensional (1D). Having new families of exactly solvable models with realistic local interactions, especially in higher dimensions, will therefore shed light on the characterization of topological phenomena and Majorana zero modes in intrinsically interacting and number-conserving systems. In addition, these results provide new Hamiltonians that can be used to experimentally realize topological states.

In this paper we take a bottom-up approach to these fundamental issues: Starting from a general BCS-type mean-field ground state $|G\rangle$, we show how to construct number-conserving parent Hamiltonians that have $|G\rangle$ as a ground state (with no approximation). This construction enables us to realize the physics of Majorana zero modes in interacting number-conserving systems in an exact manner. Following the general construction, we build specific models including a 1D Majorana double wire and a two-dimensional (2D) $p_x + ip_y$ topological superconductor, and obtain analytic expressions for the degenerate ground states in the presence of edges and vortices.

II. GENERAL CONSTRUCTION

Suppose we have an effective mean-field Hamiltonian in some BCS-like theory in any dimension with or without spin

$$\begin{aligned} K_{\mathrm{mf}} &= \sum_p \left[\xi_p a_p^\dagger a_p + \frac{1}{2}(\Delta_p^* a_{\bar{p}} a_p + \mathrm{H.c.}) \right] \\ &= \sum_p E_p \alpha_p^\dagger \alpha_p + \mathrm{const.}, \end{aligned} \tag{1}$$

where $E_p = \sqrt{\xi_p^2 + |\Delta_p|^2}$, p indexes the single-particle states (including momentum, spin, or any other quantities necessary), $\bar{p}$ denotes the time reversed state of p, and $\alpha_p = u_p a_p - v_p a_{\bar{p}}^\dagger$ are Bogoliubov quasiparticle operators with $|u_p|^2 + |v_p|^2 = 1$ and $v_p/u_p = -(E_p - \xi_p)/\Delta_p^*$. The BCS-like ground state of

*zhiyuan.wang@rice.edu

K_{mf} is (up to normalization)

$$|G_{\mathrm{BCS}}\rangle = \prod_p{}' \alpha_p \alpha_{\bar{p}} |0\rangle, \tag{2}$$

where the prime means each pair $p\bar{p}$ appears exactly once.

Our goal is to construct a number-conserving Hamiltonian whose ground state is $|G_{\mathrm{BCS}}\rangle$. To do this, we first separate each α_p into creation and annihilation parts $\alpha_p \equiv C_p - S_p^\dagger$ with $C_p = u_p a_p$ and $S_p^\dagger = v_p a_{\bar{p}}^\dagger$, and define

$$\hat{A}_{pp'} = S_p^\dagger \alpha_{p'} + S_{p'}^\dagger \alpha_p. \tag{3}$$

From $\alpha_p|G_{\mathrm{BCS}}\rangle = 0$, we know that $\hat{A}_{pp'}|G_{\mathrm{BCS}}\rangle = 0$, thus a parent Hamiltonian for $|G_{\mathrm{BCS}}\rangle$ can be constructed by

$$\hat{H} = \sum_{p_1p_2p_3p_4} H_{p_1p_2;p_3p_4} \hat{A}_{p_1p_2}^\dagger \hat{A}_{p_3p_4}, \tag{4}$$

where the matrix $H_{p_1p_2;p_3p_4}$ is required to be Hermitian. This construction suffices for $|G_{\mathrm{BCS}}\rangle$ to be a zero-energy eigenstate of $\hat{H}$. To ensure it is a ground state, we require that the matrix $H_{p_1p_2;p_3p_4}$ is positive-definite. Notice that $\hat{H}$ conserves total particle number $\hat{N} = \sum_p a_p^\dagger a_p$ since $\hat{A}_{pp'}$ can be rewritten as $\hat{A}_{pp'} = S_p^\dagger C_{p'} + S_{p'}^\dagger C_p$, which follows because $S_p^\dagger S_{p'}^\dagger + S_{p'}^\dagger S_p^\dagger$ vanishes by fermionic antisymmetry. Then the ground state of $\hat{H}$ with a definite particle number N is simply given by the projection of $|G_{\mathrm{BCS}}\rangle$ to the N-particle subspace $|G_N\rangle = \hat{P}_N|G_{\mathrm{BCS}}\rangle$.

III. THE DOUBLE WIRE MODEL

As a first specific example of this construction, we construct one-dimensional models that reproduce the ground states of Kitaev's 1D wire [1] $\hat{H}_{\mathrm{Kitaev}} = \sum_j(-tc_j^\dagger c_{j+1} + \Delta c_j c_{j+1} + \mathrm{H.c.}) - \mu\hat{N}$, where t, μ, and Δ denote the hopping amplitude, the chemical potential, and the superconducting gap, respectively. Kitaev's model has a special point at $\mu = 0, \Delta = t$ where the Hamiltonian can be rewritten as a sum of mutually commuting local operators and the spectrum is nondispersing [1]. To simplify our calculation, we focus on this special point (though our construction protocol is general and could be applied to other points as well) where the Hamiltonian has doubly degenerate ground states given by (up to normalization)

$$|G^e\rangle = \exp\left(\sum_{i<j} c_i^\dagger c_j^\dagger\right)|0\rangle,\ |G^o\rangle = \tilde{c}_{k=0}^\dagger|G^e\rangle, \tag{5}$$

where $\tilde{c}_k^\dagger = \frac{1}{\sqrt{L}}\sum_{j=1}^L e^{ikj}c_j^\dagger$. The superscripts e,o denote even and odd fermion parity, respectively.

We consider a double wire geometry that has two parallel one-dimensional chains, with fermion creation operators on each chain given by $a_j^\dagger \equiv c_{j,1}^\dagger, b_j^\dagger \equiv c_{j,2}^\dagger$, respectively. Our aim is to construct a number-conserving lattice model on these two wires whose ground states are direct products of the Kitaev ground states on each wire, projected to fixed total particle number $|G_N\rangle = \hat{P}_N(|G_A\rangle \otimes |G_B\rangle)$. We will show that the resulting Hamiltonian also leads to Majorana-like edge modes and robust ground state degeneracy.

The direct product of Kitaev ground states $|G_A\rangle \otimes |G_B\rangle$ is annihilated by Bogoliubov operators

$$\alpha_{k\sigma} = \frac{e^{i\frac{k}{2}}}{\sqrt{2}}\left(-\sin\frac{k}{2}\tilde{c}_{k\sigma} + i\cos\frac{k}{2}\tilde{c}_{-k\sigma}^\dagger\right) - (k \to -k), \tag{6}$$

where $\sigma = 1,2$, and the quasimomentum k is quantized with open boundary condition $k = \frac{m\pi}{L}, m = 1,\ldots,(L-1)$ [25].

Having identified the $\alpha_{k\sigma}$, and therefore the $C_{k\sigma}$ and $S_{k\sigma}^\dagger$, Eq. (4) gives a family of parent Hamiltonians. There are in principle infinite number of choices for $H_{p_1p_2;p_3p_4}$. However, most of these choices will lead to complicated long-range interactions. To facilitate experimental realization in cold atom systems, we are particularly interested in $H_{p_1p_2;p_3p_4}$ that lead only to spatially local interaction terms. We find one such example given by [26]

$$\begin{aligned} H_{p_1p_2;p_3p_4} = \frac{1}{L}\sum_{\xi_j=\pm1} &\xi_1\xi_2\xi_3\xi_4\delta_{\xi_1k_1+\xi_2k_2+\xi_3k_3+\xi_4k_4} \\ &\times[p\delta_{\sigma_1\sigma_2\sigma_3\sigma_4} + q\delta_{\sigma_1\sigma_2}\delta_{\sigma_3\sigma_4} \\ &- r(\delta_{\sigma_1\sigma_3}\delta_{\sigma_2\sigma_4} + \delta_{\sigma_1\sigma_4}\delta_{\sigma_2\sigma_3})], \end{aligned} \tag{7}$$

where $p_j = (k_j,\sigma_j)$ and p,q,r are arbitrary real coefficients, one obtains Hamiltonians $\hat{H}$ with interactions that are local in real space,

$$\begin{aligned} \hat{H} = \sum_{j=1}^{L-1}\Bigg\{&-t\Bigg[(a_j^\dagger a_{j+1} + b_j^\dagger b_{j+1} + \mathrm{H.c.}) - 1 + 2\left(n_j^a - \frac{1}{2}\right) \\ &\times\left(n_{j+1}^a - \frac{1}{2}\right) + 2\left(n_j^b - \frac{1}{2}\right)\left(n_{j+1}^b - \frac{1}{2}\right)\Bigg] \\ &-(\alpha J_{||,j}^\dagger J_{||,j} + \beta J_{=,j}^\dagger J_{=,j} + \gamma J_{\times,j}^\dagger J_{\times,j})\Bigg\}, \end{aligned} \tag{8}$$

where $J_{||,j} = b_ja_j - b_{j+1}a_{j+1}$, $J_{=,j} = a_{j+1}a_j - b_{j+1}b_j$, $J_{\times,j} = b_{j+1}a_j - b_ja_{j+1}$, and α,β,γ are real numbers determined by p,q,r with constraint $\alpha+\beta+\gamma = 0$. In order for the matrix Eq. (7) to be positive-definite, the α,β,γ should satisfy $\alpha < 0, \beta < t, \gamma < t$, which, combined with $\alpha+\beta+\gamma = 0$, gives the triangle region that is shown in Fig. 1. The center of the triangle $\alpha = -t, \beta = \gamma = t/2$ reproduces the model in Ref. [23]. One possible experimental realization is for $\gamma = 0$, where the proposal in Ref. [21] can be used to realize each term.

Since $\hat{H}$ preserves total particle number $N = N_A + N_B$ and single wire fermion parity $P^{A,B} = (-1)^{\hat{N}_{A,B}}$, ground states in each N-particle sector are doubly degenerate. For example, if N is even, we have (up to normalization)

$$|G_N^{ee}\rangle = \left[\sum_{i<j}(a_i^\dagger a_j^\dagger + b_i^\dagger b_j^\dagger)\right]^{N/2}|0\rangle, |G_N^{oo}\rangle = \tilde{a}_0^\dagger\tilde{b}_0^\dagger|G_{N-2}^{ee}\rangle. \tag{9}$$

This degeneracy is topologically protected in the sense that all local perturbations in the bulk, even including the ones that violate single wire parity, take the form of an identity matrix when projected to the ground state subspace. For example, using the same arguments as in Refs. [23,24], we explicitly find that the energy splitting ΔE due to perturbation $a_j^\dagger b_j + \mathrm{H.c.}$

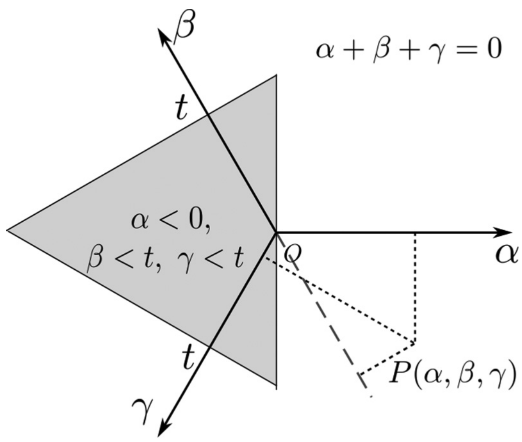

FIG. 1. Parameter region of the double wire system. The constraint $\alpha+\beta+\gamma=0$ restricts the parameter space to the 2D plane drawn in this figure, where the coordinate (α,β,γ) of an arbitrary point P is given by the projections from P to the α,β,γ axes. The conditions $\alpha<0,\beta<t,\gamma<t$, which guarantee $\hat{H}$ to be positive-definite, give a triangle region (shaded area) in this plane.

scales as $\Delta E\sim e^{-j/l_0}$ (assuming $j<L/2$) for some finite length scale l_0.

IV. THE 2D p_x+ip_y MODEL

Majorana zero modes are expected to appear at certain boundaries and in the cores of vortices in p_x+ip_y superconductors. We now show that Eq. (4) can be used to construct number-conserving parent Hamiltonians for p_x+ip_y topological superconductors that share the same ground state as the mean-field model

$$\hat{K}=\int_S\left[\frac{\nabla\psi_z^\dagger\cdot\nabla\psi_z}{2m}-(\Delta\psi_z\partial_{\bar{z}}\psi_z+\text{H.c.})+\mu\psi_z\psi_z^\dagger\right]d^2z, \tag{10}$$

where S denotes an arbitrary region in the 2D plane with complex coordinates $z=x+iy$, $\partial_z=(\partial_x-i\partial_y)/2$, $\partial_{\bar{z}}=(\partial_x+i\partial_y)/2$, $d^2z=dxdy$, and ψ_z is the fermionic annihilation operator at position z. The term $\Delta\psi_z\partial_{\bar{z}}\psi_z+\text{H.c.}$ characterizes chiral p-wave pairing, and $\mu\psi_z\psi_z^\dagger$ is the chemical potential term (which differs from the usual convention by a constant). Although in principle we can construct number-conserving parent Hamiltonians for all values of (m,Δ,μ), in the following we only consider a special point $\mu=\frac{m\Delta^2}{2}$ and use natural units $2m=m\Delta=1$ for simplicity. With an integration by parts, $\hat{K}$ can be separated into a bulk Hamiltonian and a boundary term $\hat{K}=\hat{K}_{\text{bulk}}+\hat{K}_{\text{bound}}$, with

$$\hat{K}_{\text{bulk}}=\int_S(2\partial_z\psi_z^\dagger-\psi_z)(2\partial_{\bar{z}}\psi_z-\psi_z^\dagger)d^2z,$$
$$\hat{K}_{\text{bound}}=-i\oint_{\partial S}(\psi_z^\dagger\partial_z\psi_zdz+\psi_z^\dagger\partial_{\bar{z}}\psi_zd\bar{z}), \tag{11}$$

where ∂S denotes the boundary of S. Since $\hat{K}_{\text{bulk}}$ is by construction positive-definite, the ground states of $\hat{K}$ should be annihilated by the operator $\alpha_z\equiv2\partial_{\bar{z}}\psi_z-\psi_z^\dagger$ for all $z\in S$ in order to minimize $\hat{K}_{\text{bulk}}$ (we will account for the boundary term momentarily). The ground states with even fermion parity can in general be constructed as

$$|G^e\rangle=\exp\left[\frac{1}{2}\int_S g(z,z')\psi_z^\dagger\psi_{z'}^\dagger d^2zd^2z'\right]|0\rangle, \tag{12}$$

where the two-particle wave function $g(z,z')$ satisfies $g(z,z')=-g(z',z)$ and

$$2\partial_{\bar{z}}g(z,z')=\delta^2(z-z'), \tag{13}$$

which guarantees that $\alpha_z|G^e\rangle=0$. To simultaneously minimize the boundary term $\hat{K}_{\text{bound}}$, the function $g(z,z')$ should satisfy certain boundary conditions that depend on the geometry of the region S, which we will discuss later.

A. Constructing a number-conserving parent Hamiltonian

To find a parent Hamiltonian for the mean-field ground state $|G^e\rangle$, we again follow our general construction given in Eqs. (3) and (4) where we identify $C_z=2\partial_{\bar{z}}\psi_z$ and $S_z^\dagger=\psi_z^\dagger$, leading to

$$\hat{H}_{\text{bulk}}=\int_S W(z_1,z_2;z_3,z_4)A_{z_1,z_2}^\dagger A_{z_3,z_4}\prod_{j=1}^4d^2z_j, \tag{14}$$

where $W(z_1,z_2;z_3,z_4)$ is a positive-definite Hermitian matrix. We further restrict ourselves to Hamiltonians describing short-ranged interactions, i.e., $W(z_1,z_2;z_3,z_4)$ tends to zero sufficiently fast when the distance between any two points $|z_i-z_j|$ becomes large. Furthermore, the boundary term $\hat{K}_{\text{bound}}$ in Eq. (11) should be added into $\hat{H}_{\text{bulk}}$ to uniquely pick out the same set of ground states as $\hat{K}$,

$$\hat{H}=\hat{H}_{\text{bulk}}+\hat{K}_{\text{bound}}. \tag{15}$$

The new interacting Hamiltonian $\hat{H}$ harbors the topological p_x+ip_y ground state. It is number conserving because both $\hat{H}_{\text{bulk}}$ and $\hat{K}_{\text{bound}}$ preserve total particle number.

As a specific example, we choose $W(z_1,z_2;z_3,z_4)=e^{-\lambda|z_1-z_3|}\delta^2(z_1-z_2)\delta^2(z_3-z_4)/4$ in Eq. (14) where $\lambda>0$, after rearranging terms we get

$$\hat{H}=\int_S\nabla\psi_z^\dagger\cdot\nabla\psi_zd^2z$$
$$+4\int_S e^{-\lambda|z-z'|}\psi_z^\dagger(\partial_{z'}\psi_{z'}^\dagger)\psi_{z'}\partial_{\bar{z}}\psi_zd^2zd^2z'. \tag{16}$$

It is also interesting to point out that with a different choice of the coefficient matrix $W(z_1,z_2;z_3,z_4)$ we are able to reproduce the Richardson-Gaudin p_x+ip_y model [27–30] at the "Moore-Read" line, where the model is known to be exactly solvable and exhibits a gapped spectrum in a fixed particle number sector.

B. Degenerate ground states with vortices

It is known that vortices in the mean-field model Eq. (10) have localized Majorana zero modes [12,31] giving rise to topologically protected ground state degeneracy and non-Abelian statistics. It would be interesting to see whether these important properties survive in our number-conserving model, as these properties are crucial for the realization of topological quantum computation [13]. One remarkable feature of our

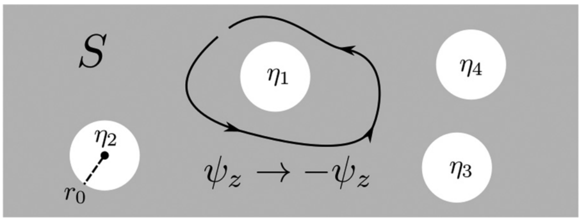

FIG. 2. Topological superconducting phase in an unbounded 2D plane with $2M$ vortices located at $\eta_1,\eta_2,\ldots,\eta_{2M}$. Here we show the $2M=4$ case. In our gauge convention fermion fields acquire a minus sign on going around each vortex.

model is that the analytic expressions of the degenerate ground states could be exactly obtained even though the vortices break translation invariance. These explicit expressions give us deeper insight into the topological properties of the ground states and manifest the non-Abelian statistics of vortices.

We consider the geometry shown in Fig. 2, with $2M$ vortices lying in an unbounded 2D plane located at $\eta_1,\eta_2,\ldots,\eta_{2M}$, respectively. We assume that the core of each vortex is localized inside a radius r_0 much smaller than the minimal distance between any two vortices. We use the gauge convention in which the superconducting order parameter is the same everywhere [i.e., still consider the same Hamiltonians in Eqs. (10) and (14)] while fermion fields are antiperiodic around each vortex $\psi^\dagger_{\theta+2\pi}=-\psi^\dagger_\theta$.

In the mean-field model Eq. (10) there is a Majorana zero-mode γ_j with $\gamma_j^\dagger=\gamma_j$ and $[\gamma_j,\hat{K}]=0$ localized at the jth vortex. In total we have $2M$ localized Majorana modes $\gamma_1,\gamma_2,\ldots,\gamma_{2M}$ which could be combined to M independent fermion operators $a_j=(\gamma_{2j-1}+i\gamma_{2j})/2$, leading to 2^M degenerate mean-field ground states $|n_1,n_2,\ldots,n_M\rangle$ with $n_j=0,1, 1\leqslant j\leqslant M$. In our number-conserving model defined in Eq. (15), the ground states with N particles are obtained by projecting the mean-field ground states to the N-particles sector. Only those mean-field states with fermion parity equal to $(-1)^N$ survive this projection, therefore we are left with 2^{M-1}-fold degeneracy in each sector.

As an example, we consider the $2M=4$ case and assume N to be even. One of the mean-field ground states $|G_{12,34}\rangle$ could be constructed by Eq. (12) with

$$g_{12,34}(z,z')=\frac{1}{4\pi(z-z')}\left[\frac{f^{12}_{34}(z)}{f^{12}_{34}(z')}+(z\leftrightarrow z')\right], \tag{17}$$

where $f^{12}_{34}(z)=\sqrt{\frac{(z-\eta_1)(z-\eta_2)}{(z-\eta_3)(z-\eta_4)}}$ is introduced to guarantee that $g_{12,34}(z,z')$ is antiperiodic around each vortex, in accordance with our gauge convention. From the identity $2\partial_{\bar{z}}\frac{1}{z-z'}=2\pi\delta^2(z-z')$ it is easy to see that $g_{12,34}(z,z')$ satisfies Eq. (13), and it can be checked that the state $|G_{12,34}\rangle$ also minimizes $\hat{K}_{\rm bound}$ up to some small corrections [32]. Applying the projection operator $\hat{P}_N$, we get an N-particle ground state with multiparticle wave function (up to normalization) $\psi_{12,34}(z_1,z_2,\ldots,z_N)=\mathrm{Pf}\{g_{12,34}(z_i,z_j)\}$, where Pf denotes the Pfaffian of the antisymmetric $N\times N$ matrix $g_{12,34}(z_i,z_j)$. The wave function $\psi_{12,34}$ is very similar (but not identical) to one of the Moore-Read Pfaffian states with four quasiholes [10,11], which were constructed to describe the excitations in the $\nu=5/2$ fractional quantum hall effect (FQHE). By permuting the indices 1,2,3,4 we get two other degenerate ground states with wave functions $\psi_{13,24}(z_1,\ldots,z_N)$ and $\psi_{14,23}(z_1,\ldots,z_N)$. However, using the same method in Ref. [11] we can prove that these three states are linearly dependent and the space spanned by them is actually two dimensional, consistent with our previous argument.

The non-Abelian statistics of the mean-field ground states of the p_x+ip_y model have been well studied in Ref. [12]. Braiding the jth and the $(j+1)$th vortices adiabatically gives rise to a unitary rotation $\hat{B}_{jj+1}=\exp(\frac{\pi}{4}\gamma_{j+1}\gamma_j)$ on the ground state subspace. It should be expected that the particle number projected ground states of our number-conserving model have the same non-Abelian statistics as the unprojected mean-field states, since the particle number fluctuations of the mean-field states are negligible in the thermodynamic limit, and the Berry's matrices of the braiding process should be the same. The rigorous proof of this argument will be the subject of future work.

V. PHASE DIAGRAM AND EXCITATION SPECTRUM

We now briefly discuss to what extent does our construction gives gapped phases of matter and comment on the low energy excitations of our models.

All models constructed by Eq. (4) sit on a critical point between a phase separated state (upon adding attractive density-density interactions) and, potentially, a topological superconducting state (repulsive interactions). In the double wire model for example, right at the critical point, the phase separation state $|E^{\rm PS}_{N=L}\rangle=\prod_{j\leqslant L/2}a_j^\dagger b_j^\dagger|0\rangle$ has a finite excitation energy Δ (the energy of the domain wall, independent of system size L), if a small nearest-neighbor interaction $-v\sum_j(n^a_jn^a_{j+1}+n^b_jn^b_{j+1})$ is added, the energy of $|E^{\rm PS}_{N=L}\rangle$ would be $\Delta-v(L-2)$, while the energy of the homogeneous superconducting ground state is approximately $-v(L-1)/2$. Thus when $L\to\infty$, with an infinitesimal attraction $v>0$ the phase separation state would be favored, while for repulsive interaction $v\leqslant 0$ the topological superconducting state has lower energy. This was also numerically verified for a special case of the double wire model in Refs. [23,33].

There are gapless Goldstone excitations in the double wire model Eq. (8) and the p_x+ip_y model Eq. (16) provided that $w(z-z')$ is sufficiently short ranged, while the exponential decay of the one-particle correlation function indicates that single particle excitation is gapped in both models [23]. The presence of Goldstone excitation is a universal feature of neutral topological superconductor [13], which belongs to the gapless symmetry-protected topological phase [34,35]. In charged superconductors however, the Anderson-Higgs mechanism would lift the Goldstone mode way up to the plasma frequency and the system would have a gapped excitation spectrum. This means that we can gap the Goldstone mode in our current construction by either coupling our system to quantized electromagnetic gauge field or allowing long-range interactions, an extreme example is that a special case of Eq. (14) reproduce the Richardson-Gaudin models [27–30] at the "Moore-Read" line, whose exact solution shows a gapped spectrum.

VI. CONCLUSION

We have constructed infinite families of number-conserving, interacting Hamiltonians with exact BCS-like ground states, with specific models including a 1D Majorana double wire and a 2D $p_x + ip_y$ topological superconductor. In the $p_x + ip_y$ model we obtained analytic expressions of degenerate ground states with four vortices, and pointed out their similarity to the Moore-Read Pfaffian states with four quasiholes constructed in the $\nu = 5/2$ FQHE context. Our models give us a deeper theoretical understanding of topological phenomena in interacting systems, set a viable framework for building more realistic models of topological superconductors, and may provide useful guidelines for experimental realization of Majorana zero modes.

ACKNOWLEDGMENTS

We thank Matthew Foster and Bhuvanesh Sundar for discussions. H.P. was supported by the NSF and the Welch Foundation (Grant No. C-1669). K.R.A.H. was supported in part with funds from the Welch Foundation (Grant No. C-1872).

APPENDIX A: THE DOUBLE WIRE MODEL

In this Appendix we first present the detailed derivations of Eqs. (5)–(8) in the main text, and then we discuss some alternative derivations of the parent Hamiltonian.

1. Diagonalization of Kitaev's Hamiltonian in momentum space with open boundary

To obtain the Bogoliubov operators in Eq. (6) and the form of ground states in Eq. (5) in our main text, here we present the momentum space diagonalization of Kitaev's Hamiltonian with $\Delta = t$,

$$H_{\text{Kitaev}} = \sum_j t(-c_j^\dagger c_{j+1} + c_j c_{j+1} + \text{H.c.}) - \mu \hat{N}. \quad \text{(A1)}$$

To this end we search for Bogoliubov eigenmodes defined as

$$\alpha_k = \sum_{j=1}^{L} (u_j^k c_j + v_j^k c_j^\dagger). \quad \text{(A2)}$$

Being the eigenmodes of H_{Kitaev} with energy $E_k > 0$, they satisfy $[\alpha_k, H_{\text{Kitaev}}] = E_k \alpha_k$, which gives difference equations on u_j^k, v_j^k,

$$\begin{aligned}(E_k + \mu)u_j^k &= -t(u_{j-1}^k + u_{j+1}^k) + t(v_{j-1}^k - v_{j+1}^k),\\ (E_k - \mu)v_j^k &= t(v_{j-1}^k + v_{j+1}^k) + t(u_{j+1}^k - u_{j-1}^k),\end{aligned} \quad \text{(A3)}$$

with boundary conditions

$$u_0^k = v_0^k, u_{L+1}^k = -v_{L+1}^k. \quad \text{(A4)}$$

To solve these equations, we notice that the ansatz solutions

$$\begin{aligned}u_j^k &= \lambda(k)e^{i(kj-\theta_k)} - \lambda(-k)e^{-i(kj-\theta_k)},\\ v_j^k &= e^{i(kj-\theta_k)} - e^{-i(kj-\theta_k)},\end{aligned} \quad \text{(A5)}$$

with $\lambda(k) = -i\frac{2\Delta \sin k}{E_k + \mu + 2t\cos k}$ and $E_k = \sqrt{\mu^2 + 4t\mu\cos k + 4t^2}$ satisfy Eq. (A3). The boundary conditions in Eq. (A4) give constraints on the quasimomentum k and the real parameter θ_k,

$$\begin{aligned}k(L+1) &= 2\theta_k + m\pi, \quad m \in \mathbf{Z},\\ \tan\theta_k &= \frac{2\Delta \sin k}{E_k + \mu + 2t\cos k},\end{aligned} \quad \text{(A6)}$$

where $0 < k < \pi$ and $0 < \theta_k < \pi/2$. The Bogoliubov operators are (up to normalization)

$$\alpha_k = e^{i\theta_k}(i\cos\theta_k c_{-k}^\dagger - \sin\theta_k c_k) - (k \to -k). \quad \text{(A7)}$$

As an aside, we mention that if we replace Eq. (6) in our main text by Eq. (A7) and use the same matrix in Eq. (7), then, still following our general construction, we can get a bigger family of number-conserving, short-range interacting Hamiltonians with ground states $|G^{e,o}\rangle$ depending on μ/t, and this method can be generalized to construct parent Hamiltonians for Kitaev's ground states at arbitrary points (t,Δ,μ) (even including points in the topologically trivial phase).

At the $\mu = 0$ point we get especially simple expressions

$$E_k = 2t, \quad \theta_k = \frac{k}{2}, \quad k = \frac{m\pi}{L}, \quad m = 1,\ldots,(L-1), \quad \text{(A8)}$$

which leads to the single wire version of Bogoliubov operators in Eq. (6) after normalization. To verify that the expressions given in Eq. (5) are indeed the ground states of H_{Kitaev}, we show that $|G^e\rangle$ and $|G^o\rangle$ are annihilated by all α_k. We have

$$\begin{aligned}c_k|G^e\rangle &= \frac{1}{\sqrt{L}}\sum_{j=1}^{L} e^{-ikj}c_j \exp\left\{\sum_{i<j'} c_i^\dagger c_{j'}^\dagger\right\}|0\rangle\\ &= \frac{1}{\sqrt{L}}\sum_{j,j'} e^{-ikj}\,\text{sgn}(j'-j)c_{j'}^\dagger|G^e\rangle\\ &= \left[i\cot\frac{k}{2}c_{-k}^\dagger - \frac{1+(-1)^m}{1-e^{ik}}c_{k=0}^\dagger\right]|G^e\rangle,\end{aligned} \quad \text{(A9)}$$

where $\text{sgn}(x) = x/|x|$ for $x \neq 0$ and $\text{sgn}(0) = 0$. It follows that

$$\begin{aligned}&\left[e^{i\frac{k}{2}}\sin\frac{k}{2}c_k - (k \to -k)\right]|G^e\rangle\\ &\quad = \left[e^{i\frac{k}{2}}i\cos\frac{k}{2}c_{-k}^\dagger - (k \to -k)\right]|G^e\rangle,\end{aligned} \quad \text{(A10)}$$

leading to $\alpha_k|G^e\rangle = 0$. Furthermore, it can be easily checked that $\{c_{k=0}^\dagger, \alpha_k\} = 0$ for all $k = m\pi/L, 1 \leqslant m \leqslant L-1$, thus $\alpha_k|G^o\rangle = \alpha_k c_{k=0}^\dagger|G^e\rangle = -c_{k=0}^\dagger \alpha_k|G^e\rangle = 0$. We conclude that Eq. (5) indeed gives us the ground states of H_{Kitaev} at the point $t = \Delta, \mu = 0$.

2. Detailed derivation of Eq. (8)

To verify that the combination of Eqs. (4) and (7) indeed give the local form of Eq. (8), we first notice that

$$\begin{aligned}\alpha_{k\sigma} &\equiv C_{k\sigma} - S^\dagger_{k\sigma} \\ &= \frac{e^{i\frac{k}{2}}}{\sqrt{2}}\left(i\cos\frac{k}{2}c^\dagger_{-k,\sigma} - \sin\frac{k}{2}c_{k\sigma}\right) - (k\to -k) \\ &= \frac{1}{\sqrt{2L}}\sum_{j=1}^{L-1}\sin kj(c_{j+1,\sigma} + c^\dagger_{j+1,\sigma} - c_{j\sigma} + c^\dagger_{j\sigma}),\end{aligned} \tag{A11}$$

where $k = \frac{m\pi}{L}, m = 1,\ldots,(L-1)$. Using the completeness and orthonormality of $\sin kj$,

$$\sum_k \sin kj \sin kj' = \frac{L}{2}\delta_{jj'}, \tag{A12}$$

we have

$$\begin{aligned}&\frac{4}{\sqrt{2L}}\sum_k (C_{k\sigma} - S^\dagger_{k\sigma})\sin kj \\ &= c_{j+1,\sigma} + c^\dagger_{j+1,\sigma} - c_{j\sigma} + c^\dagger_{j\sigma} \\ &\equiv C_{j\sigma} - S^\dagger_{j\sigma}, \quad 1 \leqslant j \leqslant L-1,\end{aligned} \tag{A13}$$

where $C_{j\sigma} = c_{j+1,\sigma} - c_{j\sigma}, S^\dagger_{j\sigma} = -c^\dagger_{j+1,\sigma} - c^\dagger_{j\sigma}$. By applying the identity

$$\begin{aligned}&\sum_{\xi_j=\pm 1}\xi_1\xi_2\xi_3\xi_4\delta_{\xi_1k_1+\xi_2k_2+\xi_3k_3+\xi_4k_4} \\ &= \frac{16}{L}\sum_{j=1}^{L-1}\sin k_1 j \sin k_2 j \sin k_3 j \sin k_4 j,\end{aligned} \tag{A14}$$

the parent Hamiltonian given in Eqs. (4) and (7) can then be expanded in position space (we use the shorthand $\sum_{\boldsymbol{k}} = \sum_{k_1,k_2,k_3,k_4}$ and $\sum_{\boldsymbol{\sigma}} = \sum_{\sigma_1,\sigma_2,\sigma_3,\sigma_4}$)

$$\begin{aligned}H &= \sum_{p_1p_2p_3p_4} H_{p_1p_2;p_3p_4}\hat{A}^\dagger_{p_1p_2}\hat{A}_{p_3p_4} = \frac{16}{L^2}\sum_{\boldsymbol{k},\boldsymbol{\sigma}}\sum_{j=1}^{L-1}\sin k_1 j \sin k_2 j \sin k_3 j \sin k_4 j \\ &\quad\times\left[p\delta_{\sigma_1\sigma_2\sigma_3\sigma_4} + q\delta_{\sigma_1\sigma_2}\delta_{\sigma_3\sigma_4} - r(\delta_{\sigma_1\sigma_3}\delta_{\sigma_2\sigma_4} + \delta_{\sigma_1\sigma_4}\delta_{\sigma_2\sigma_3})\right](C^\dagger_{k_1\sigma_1}S_{k_2\sigma_2} + C^\dagger_{k_2\sigma_2}S_{k_1\sigma_1})(S^\dagger_{k_3\sigma_3}C_{k_4\sigma_4} + S^\dagger_{k_4\sigma_4}C_{k_3\sigma_3}) \\ &= \frac{64}{L^2}\sum_{\boldsymbol{k},\boldsymbol{\sigma}}\sum_{j=1}^{L-1}(C^\dagger_{k_1\sigma_1}\sin k_1 j)(S_{k_2\sigma_2}\sin k_2 j)(S^\dagger_{k_3\sigma_3}\sin k_3 j)(C_{k_4\sigma_4}\sin k_4 j) \\ &\quad\times\left[p\delta_{\sigma_1\sigma_2\sigma_3\sigma_4} + q\delta_{\sigma_1\sigma_2}\delta_{\sigma_3\sigma_4} - r(\delta_{\sigma_1\sigma_3}\delta_{\sigma_2\sigma_4} + \delta_{\sigma_1\sigma_4}\delta_{\sigma_2\sigma_3})\right] \\ &= \sum_{\boldsymbol{\sigma}}\sum_{j=1}^{L-1} C^\dagger_{j\sigma_1}S_{j\sigma_2}S^\dagger_{j\sigma_3}C_{j\sigma_4}\left[p\delta_{\sigma_1\sigma_2\sigma_3\sigma_4} + q\delta_{\sigma_1\sigma_2}\delta_{\sigma_3\sigma_4} - r(\delta_{\sigma_1\sigma_3}\delta_{\sigma_2\sigma_4} + \delta_{\sigma_1\sigma_4}\delta_{\sigma_2\sigma_3})\right] \\ &= (2t-\beta-\gamma)\sum_{j=1}^{L-1}[C^\dagger_{aj}S_{aj}S^\dagger_{aj}C_{aj} + (a\to b)] + (\gamma-\beta)\sum_{j=1}^{L-1}[C^\dagger_{aj}S_{aj}S^\dagger_{bj}C_{bj} + (a\leftrightarrow b)] \\ &\quad+(\beta+\gamma)\sum_{j=1}^{L-1}(C^\dagger_{aj}S_{bj} + C^\dagger_{bj}S_{aj})(S^\dagger_{aj}C_{bj} + S^\dagger_{bj}C_{aj}),\end{aligned} \tag{A15}$$

where $\beta = -(q+r)/2, \gamma = (q-r)/2$ and $t = (p+q-3r)/2$. By expanding the last line of Eq. (A15) we get the form of Eq. (8) in the main text (with $\alpha = r = -\beta-\gamma$).

3. Positive region of $\hat{H}$

We now prove that the matrix $H_{p_1p_2;p_3p_4}$ given in Eq. (7) is positive-definite in the triangle region shown in Fig. 1 in the main text. Notice that $H_{p_1p_2;p_3p_4} = H_{k_1k_2;k_3k_4}\lambda_{\sigma_1\sigma_2;\sigma_3\sigma_4}$ with orbital part $H_{k_1k_2;k_3k_4} = \frac{16}{L^2}\sum_{j=1}^{L}\sin k_1 j \sin k_2 j \sin k_3 j \sin k_4 j$ [see Eq. (A14)] and spin part $\lambda_{\sigma_1\sigma_2;\sigma_3\sigma_4} = p\delta_{\sigma_1\sigma_2\sigma_3\sigma_4} + q\delta_{\sigma_1\sigma_2}\delta_{\sigma_3\sigma_4} - r(\delta_{\sigma_1\sigma_3}\delta_{\sigma_2\sigma_4} + \delta_{\sigma_1\sigma_4}\delta_{\sigma_2\sigma_3})$. The orbital part is always positive, since for any vector $f_{kk'}$ we have

$$\sum_{k_1k_2k_3k_4} f^*_{k_1k_2}H_{k_1k_2;k_3k_4}f_{k_3k_4} = \frac{16}{L^2}\sum_{k_1k_2k_3k_4}\sum_{j=1}^{L} f^*_{k_1k_2}\sin k_1 j \sin k_2 j \sin k_3 j \sin k_4 j f_{k_3k_4} = \frac{16}{L^2}\sum_{j=1}^{L} f^*_j f_j \geqslant 0, \tag{A16}$$

where $f_j = \sum_{kk'} f_{kk'}\sin kj \sin k'j$. For the spin part $\lambda_{\sigma_1\sigma_2;\sigma_3\sigma_4}$, we write it in the matrix form (assume the order aa,ab,ba,bb)

$$\lambda = \begin{bmatrix} p+q-2r & 0 & 0 & q \\ 0 & -r & -r & 0 \\ 0 & -r & -r & 0 \\ q & 0 & 0 & p+q-2r\end{bmatrix} = \begin{bmatrix} 2t-\gamma-\beta & 0 & 0 & \gamma-\beta \\ 0 & -\alpha & -\alpha & 0 \\ 0 & -\alpha & -\alpha & 0 \\ \gamma-\beta & 0 & 0 & 2t-\gamma-\beta\end{bmatrix} = \lambda_1 \oplus \lambda_2, \tag{A17}$$

with $\lambda_1 = \begin{bmatrix} -\alpha & -\alpha \\ -\alpha & -\alpha \end{bmatrix}$ acting on (ab,ba) and $\lambda_2 = \begin{bmatrix} 2t-\gamma-\beta & \gamma-\beta \\ -\gamma-\beta & 2t-\gamma-\beta \end{bmatrix}$ acting on (aa,bb). Thus λ is positive-definite if and only if both λ_1 and λ_2 are positive-definite. The condition that λ_1 is positive-definite gives $-\alpha > 0$, while λ_2 is positive-definite gives $|\gamma - \beta| < 2t - \gamma - \beta$, which simplifies to $\alpha < 0, \beta < t, \gamma < t$, leading to the triangle region in the main text.

4. Alternative derivations of the parent Hamiltonian

The derivation of $\hat{H}$ presented above enables us to see how the double wire model follows from our general construction and can be generalized to arbitrary points of Kitaev's model. However, for the double wire parent Hamiltonian constructed in our main text, simpler derivations exist. Actually, Eq. (A13) gives us annihilators of the double wire ground state $|G_A\rangle \otimes |G_B\rangle$ in position space. Thus we can directly build the parent Hamiltonian $\hat{H}$ using the real space version of Eqs. (3) and (4) with $C_{j\sigma} = c_{j+1,\sigma} - c_{j\sigma}, S^\dagger_{j\sigma} = -c^\dagger_{j+1,\sigma} - c^\dagger_{j\sigma}$ [or equivalently, directly go to the last line of Eq. (A15) without working in momentum space at all], leading to the same Hamiltonian Eq. (8) in our main text. This derivation is a direct generalization of the one given in Ref. [23].

Another simple derivation is based on using an alternative basis of single wire ground states (at $\Delta = t, \mu = 0$) [36]

$$|G^\eta\rangle = (1 + \eta c_1^\dagger)(1 + \eta c_2^\dagger)\cdots(1 + \eta c_L^\dagger)|0\rangle, \quad \eta = \pm 1, \tag{A18}$$

and observing the following properties:

$$\begin{aligned} c_i^\dagger c_{i+1}^\dagger |G^\eta\rangle &= n_i n_{i+1} |G^\eta\rangle, \\ c_{i+1} c_i |G^\eta\rangle &= \bar{n}_i \bar{n}_{i+1} |G^\eta\rangle, \\ c_i^\dagger c_{i+1} |G^\eta\rangle &= n_i \bar{n}_{i+1} |G^\eta\rangle, \\ c_{i+1}^\dagger c_i |G^\eta\rangle &= \bar{n}_i n_{i+1} |G^\eta\rangle, \end{aligned} \tag{A19}$$

where $\bar{n}_i \equiv 1 - n_i$. With this, it is easy to check that the number-conserving single wire operator $[(c_j^\dagger c_{j+1} + \text{H.c.}) + 2(n_j - \frac{1}{2})(n_{j+1} - \frac{1}{2}) - \frac{1}{2}]$ annihilates $|G^\eta\rangle$. To include interwire couplings, we notice that

$$\begin{aligned} J^\dagger_{||,j} J_{||,j} |G^{\eta_A,\eta_B}\rangle &= J^\dagger_{=,j} J_{=,j} |G^{\eta_A,\eta_B}\rangle \\ = J^\dagger_{\times,j} J_{\times,j} |G^{\eta_A,\eta_B}\rangle &= (U_j^\square - U_j^{(3p)}) |G^{\eta_A,\eta_B}\rangle, \end{aligned} \tag{A20}$$

where $\hat{U}_j^\square = n_j^a n_{j+1}^a n_j^b n_{j+1}^b$, $\hat{U}_j^{(3p)} = [n_j^a n_{j+1}^a (n_j^b + n_{j+1}^b) + (a \leftrightarrow b)]$, and $|G^{\eta_A,\eta_B}\rangle = |G_A^{\eta_A}\rangle \otimes |G_B^{\eta_B}\rangle$ is the double wire ground state constructed by direct product of single wire ground states. Therefore,

$$(\alpha J^\dagger_{||,j} J_{||,j} + \beta J^\dagger_{=,j} J_{=,j} + \gamma J^\dagger_{\times,j} J_{\times,j}) |G^{\eta_A,\eta_B}\rangle = 0, \tag{A21}$$

for $\alpha + \beta + \gamma = 0$. It then follows that the Hamiltonian constructed in Eq. (8) in the main text satisfies $\hat{H}|G^{\eta_A,\eta_B}\rangle = 0$, i.e., $|G^{\eta_A,\eta_B}\rangle$ is an eigenstate of $\hat{H}$ with zero energy. This method can only tell us that $|G^{\eta_A,\eta_B}\rangle$ is an eigenstate of $\hat{H}$. To find the positive region of $\hat{H}$ (where $|G^{\eta_A,\eta_B}\rangle$ become its ground states), we still have to turn to other means.

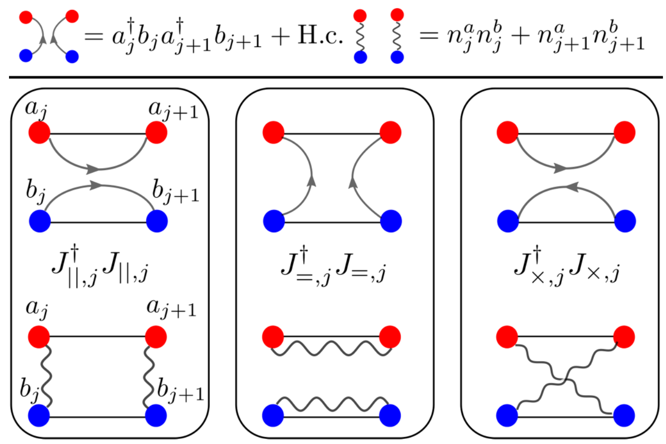

FIG. 3. Diagrammatic representation of interaction terms in the double wire parent Hamiltonian in Eq. (8). Double arrows represent pair hopping while wavy lines represent interactions.

In Fig. 3 we draw a pictorial representation of the interaction terms of the parent Hamiltonian Eq. (8), including two types of nonlinear terms: interactions and correlated pair tunnelings.

APPENDIX B: RELATION OF OUR CONSTRUCTION TO THE RICHARDSON-GAUDIN MODELS

As mentioned in the main text, by suitably choosing the coefficient matrix in Eq. (14) we are able to reproduce the Richardson-Gaudin $p_x + ip_y$ model [27–29] at the Moore-Read line. In this section we present this relation in detail.

The Richardson-Gaudin $p_x + ip_y$ model is defined by the Hamiltonian

$$\hat{H}_{\text{RG}} = \hat{H}_0 - G b^\dagger b, \tag{B1}$$

where $G > 0$ is a coupling constant and

$$\begin{aligned} \hat{H}_0 &= \sum_k \frac{|k|^2}{2} \psi_k^\dagger \psi_k, \quad \hat{N} = \sum_k \psi_k^\dagger \psi_k, \\ b^\dagger &= \frac{1}{2} \sum_k (k_x - ik_y) \psi_k^\dagger \psi_{-k}^\dagger, \end{aligned} \tag{B2}$$

where we use complex coordinates $k = k_x + ik_y, \bar{k} = k_x - ik_y$ to denote momentum (for simplicity we use antiperiodic boundary condition in this section to avoid subtleties with the $k = 0$ mode). At the Moore-Read line $1/G = L + 1 - M$ [where M is a positive integer and L is the total number of $(k, -k)$ levels], it has been found [27–29] that the ground state is exactly the Moore-Read Pfaffian state with M Cooper pairs

$$|\psi_M\rangle = \left(\frac{1}{2} \sum_k \frac{1}{k_x + ik_y} \psi_k^\dagger \psi_{-k}^\dagger \right)^M |0\rangle. \tag{B3}$$

The state $|\psi_M\rangle$ turns out to coincide with the projected mean-field ground state $|\psi_M\rangle = \hat{P}_{N=2M}|G^e\rangle$, where $|G^e\rangle$ is the mean-field ground state given in Eq. (12) in our main text, and $g_k = 1/(k_x + ik_y)$ is exactly the solution to Eq. (13)

in momentum space with antiperiodic boundary condition. By our construction we know that $|\psi_M\rangle$ is the exact $2M$-particle ground state of the number-conserving Hamiltonian Eq. (14) and it should therefore be expected that Eq. (14) includes $\hat{H}_{\mathrm{RG}}$ as a special case. To prove this, we rewrite Eq. (14) in momentum space where the annihilators are $A_{kk'} = \psi^\dagger_{-k}k'\psi_{k'} + \psi^\dagger_{-k'}k\psi_k$ and the parent Hamiltonian is

$$\hat{H} = \sum_{k_1k_2k_3k_4} W(k_1,k_2;k_3,k_4)(\bar{k}_1\psi^\dagger_{k_1}\psi_{-k_2} + \bar{k}_2\psi^\dagger_{k_2}\psi_{-k_1}) \times(\psi^\dagger_{-k_3}k_4\psi_{k_4} + \psi^\dagger_{-k_4}k_3\psi_{k_3}). \tag{B4}$$

We choose $W(k_1,k_2;k_3,k_4) = \delta_{k_1k_3}\delta_{k_2k_4}/8$, and after normal ordering, we get

$$\hat{H} = \hat{H}_0\left(L+1-\frac{\hat{N}}{2}\right) - b^\dagger b. \tag{B5}$$

Notice that $(L+1-\frac{\hat{N}}{2})$ is a positive-definite, invertible operator that commutes with $\hat{H}$. Thus we can redefine

$$\hat{H}'_{\mathrm{RG}} \equiv \frac{1}{L+1-\frac{\hat{N}}{2}}\hat{H} = \hat{H}_0 - \frac{1}{L+1-\frac{\hat{N}}{2}}b^\dagger b. \tag{B6}$$

Comparing with Eq. (B1) we find that $\hat{H}'_{\mathrm{RG}}$ and $\hat{H}_{\mathrm{RG}}$ are exactly the same when restricted to a fixed particle number sector $N = 2M$.

As an aside, we point out that in the 1D case, our construction can also reproduce the Richardson-Gaudin-Kitaev chain [30] at the Moore-Read line

$$\hat{H}_{\mathrm{RGK}} = \frac{1}{2}\sum_k \sin^2(k/2)c^\dagger_k c_k - \frac{1}{L+1-M}\frac{1}{4}\sum_{k,k'}\sin\frac{k}{2}\sin\frac{k'}{2}c^\dagger_k c^\dagger_{-k}c_{-k'}c_{k'}, \tag{B7}$$

if we do the similar calculation as in Eqs. (B4)–(B6) for the 1D ground state

$$|\psi_M\rangle = \left(\frac{1}{2}\sum_k \frac{1}{\sin\frac{k}{2}}c^\dagger_k c^\dagger_{-k}\right)^M|0\rangle. \tag{B8}$$

APPENDIX C: CALCULATION OF THE SINGLE PARTICLE CORRELATION FUNCTION FOR THE $p_x + ip_y$ MODEL

In this Appendix we calculate the single particle correlation function of the particle-number projected ground states

$$\langle G_{2M+1}|\psi^\dagger_{\mathbf{r}}\psi_{\mathbf{r}'}|G_{2M+1}\rangle = \frac{1}{L^2}\sum_{\mathbf{k}} e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\langle G_{2M+1}|\psi^\dagger_{\mathbf{k}}\psi_{\mathbf{k}}|G_{2M+1}\rangle, \tag{C1}$$

where for simplicity we use periodic boundary condition and $|G_{2M+1}\rangle$ is the unique ground state with $2M+1$ particles, L is the system size and M is the number of Cooper pairs. Notice that the projected ground state can be represented as

$$|G_{2M+1}\rangle = C^{-1/2}\hat{P}_{2M+1}|G\rangle = C^{-1/2}\frac{1}{2\pi i}\oint_{|\xi|=1} d\xi\frac{\xi^{\hat{N}}}{\xi^{2M+2}}|G\rangle, \tag{C2}$$

where $\hat{N}$ is the particle number operator, C is a normalization factor, and

$$|G\rangle = \psi^\dagger_{k=0}\prod_k{}' \frac{(k+\psi^\dagger_k\psi^\dagger_{-k})}{\sqrt{|k|^2+1}}|0\rangle \tag{C3}$$

is the mean-field ground state, where $k = k_x + ik_y$ and the prime means each pair $(k,-k)$ appears exactly once. Thus

$$\langle G_{2M+1}|\psi^\dagger_{\mathbf{k}}\psi_{\mathbf{k}}|G_{2M+1}\rangle = \frac{\langle G|\psi^\dagger_{\mathbf{k}}\psi_{\mathbf{k}}\hat{P}_{2M+1}|G\rangle}{\langle G|\hat{P}_{2M+1}|G\rangle} = \left[\oint_{|\xi|=1}\frac{1}{\xi^{2M+1}}\frac{\xi^2}{|k|^2+\xi^2}g(\xi)d\xi\right]\Big/ \times\left[\oint_{|\xi|=1}\frac{1}{\xi^{2M+1}}g(\xi)d\xi\right], \tag{C4}$$

where

$$g(\xi) = \prod_k{}'(|k|^2+\xi^2) = \exp\left[\sum_k{}'\ln(|k|^2+\xi^2)\right] \approx \exp\left[\frac{L^2}{2(2\pi)^2}\int_{|k|<\Lambda}\ln(|k|^2+\xi^2)d^2k\right] = \exp\frac{L^2}{8\pi}\left[(\Lambda^2+\xi^2)\ln(\Lambda^2+\xi^2) - \xi^2\ln\xi^2\right], \tag{C5}$$

where we have introduced an ultraviolet momentum cutoff $|k| < \Lambda$. The total number of states is therefore $\Omega = \pi\Lambda^2/(\frac{2\pi}{L})^2$, and the filling fraction is defined as $\nu = \frac{2M+1}{\Omega} = \frac{4\pi(2M+1)}{\Lambda^2L^2}$. The integrals of Eq. (C4) are of the form $\oint f(\xi)e^{h(\xi)}d\xi$, where $h(\xi) = \ln g(\xi) - (2M+1)\ln\xi$ and can be calculated using the saddle point approximation in the thermodynamic limit: $M\to\infty$, Λ,ν fixed. The result is

$$\langle G_{2M+1}|\psi^\dagger_{\mathbf{k}}\psi_{\mathbf{k}}|G_{2M+1}\rangle = \frac{\xi_0^2}{|k|^2+\xi_0^2} + O\left(\frac{1}{M}\right), \tag{C6}$$

where ξ_0 is the saddle point of the exponent $h(\xi)$ satisfying $h'(\xi_0) = 0$, which can be simplified to

$$\ln\left(1+\frac{\Lambda^2}{\xi_0^2}\right) = \nu\frac{\Lambda^2}{\xi_0^2}. \tag{C7}$$

Substituting Eq. (C6) into Eq. (C1), we get (in the thermodynamic limit)

$$\langle G_{2M+1}|\psi^\dagger_{\mathbf{r}}\psi_{\mathbf{r}'}|G_{2M+1}\rangle = \frac{1}{(2\pi)^2}\int d^2k\frac{\xi_0^2}{|\mathbf{k}|^2+\xi_0^2}e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} = \frac{\xi_0^2}{2\pi}K_0(\xi_0|\mathbf{r}-\mathbf{r}'|), \tag{C8}$$

where $K_0(x)$ is the modified Bessel function of the second kind. Since $K_0(x)\approx\sqrt{\frac{\pi}{2x}}e^{-x}$ as $x\to\infty$,

$\langle G_{2M+1}|\psi^\dagger_{\mathbf{r}}\psi_{\mathbf{r}'}|G_{2M+1}\rangle$ decays exponentially fast as $|\mathbf{r}-\mathbf{r}'|\to\infty$. The exponential decay of the single particle correlation function suggests that single particle excitation spectrum is gapped [37].

[1] A. Y. Kitaev, Phys. Usp. **44**, 131 (2001).
[2] X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. **83**, 1057 (2011).
[3] X.-G. Wen, Int. J. Mod. Phys. B **4**, 239 (1990).
[4] A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, Phys. Rev. B **78**, 195125 (2008).
[5] X.-L. Qi, T. L. Hughes, and S.-C. Zhang, Phys. Rev. B **78**, 195424 (2008).
[6] A. Kitaev, AIP Conf. Proc. **1134**, 22 (2009).
[7] X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B **83**, 035107 (2011).
[8] X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B **82**, 155138 (2010).
[9] N. Schuch, D. Pérez-García, and I. Cirac, Phys. Rev. B **84**, 165139 (2011).
[10] G. Moore and N. Read, Nucl. Phys. B **360**, 362 (1991).
[11] C. Nayak and F. Wilczek, Nucl. Phys. B **479**, 529 (1996).
[12] D. A. Ivanov, Phys. Rev. Lett. **86**, 268 (2001).
[13] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma, Rev. Mod. Phys. **80**, 1083 (2008).
[14] S. Nadj-Perge, I. K. Drozdov, J. Li, H. Chen, S. Jeon, J. Seo, A. H. MacDonald, B. A. Bernevig, and A. Yazdani, Science **346**, 602 (2014).
[15] J.-P. Xu, M.-X. Wang, Z. L. Liu, J.-F. Ge, X. Yang, C. Liu, Z. A. Xu, D. Guan, C. L. Gao, D. Qian *et al.*, Phys. Rev. Lett. **114**, 017001 (2015).
[16] S. M. Albrecht, A. Higginbotham, M. Madsen, F. Kuemmeth, T. S. Jespersen, J. Nygård, P. Krogstrup, and C. M. Marcus, Nature (London) **531**, 206 (2016).
[17] H.-H. Sun, K.-W. Zhang, L.-H. Hu, C. Li, G.-Y. Wang, H.-Y. Ma, Z.-A. Xu, C.-L. Gao, D.-D. Guan, Y.-Y. Li *et al.*, Phys. Rev. Lett. **116**, 257003 (2016).
[18] L. Fidkowski, R. M. Lutchyn, C. Nayak, and M. P. A. Fisher, Phys. Rev. B **84**, 195436 (2011).
[19] J. D. Sau, B. I. Halperin, K. Flensberg, and S. Das Sarma, Phys. Rev. B **84**, 144509 (2011).
[20] M. Cheng and H.-H. Tu, Phys. Rev. B **84**, 094503 (2011).
[21] C. V. Kraus, M. Dalmonte, M. A. Baranov, A. M. Läuchli, and P. Zoller, Phys. Rev. Lett. **111**, 173004 (2013).
[22] H. Katsura, D. Schuricht, and M. Takahashi, Phys. Rev. B **92**, 115137 (2015).
[23] F. Iemini, L. Mazza, D. Rossini, R. Fazio, and S. Diehl, Phys. Rev. Lett. **115**, 156402 (2015).
[24] N. Lang and H. P. Büchler, Phys. Rev. B **92**, 041118(R) (2015).
[25] Notice that with open boundary condition $k = m\pi/L$, the $\tilde{c}_{k\sigma},\tilde{c}^\dagger_{k'\sigma'}$ no longer satisfy the canonical anticommutation relations, but $\{\alpha_{k\sigma},\alpha^\dagger_{k'\sigma'}\} = \delta_{kk'}\delta_{\sigma\sigma'}$.
[26] Here the Kronecker delta function δ_k is defined to be periodic with period 2π, i.e., $\delta_k = 1$ if $k \equiv 0 \pmod{2\pi}$ and $\delta_k = 0$ otherwise.
[27] M. Ibañez, J. Links, G. Sierra, and S.-Y. Zhao, Phys. Rev. B **79**, 180501(R) (2009).
[28] S. M. A. Rombouts, J. Dukelsky, and G. Ortiz, Phys. Rev. B **82**, 224510 (2010).
[29] C. Dunning, M. Ibanez, J. Links, G. Sierra, and S.-Y. Zhao, J. Stat. Mech. (2010) P08025.
[30] G. Ortiz, J. Dukelsky, E. Cobanera, C. Esebbag, and C. Beenakker, Phys. Rev. Lett. **113**, 267002 (2014).
[31] N. Read and D. Green, Phys. Rev. B **61**, 10267 (2000).
[32] Z. Wang and K. R. A. Hazzard (unpublished).
[33] K. Guther, N. Lang, and H. P. Büchler, arXiv:1705.01786.
[34] A. Keselman and E. Berg, Phys. Rev. B **91**, 235309 (2015).
[35] P. Bonderson and C. Nayak, Phys. Rev. B **87**, 195451 (2013).
[36] M. Greiter, V. Schnells, and R. Thomale, Ann. Phys. **351**, 1026 (2014).
[37] M. B. Hastings and T. Koma, Commun. Math. Phys. **265**, 781 (2006).